A solution method for general contact–impact problems

K. Farahani, M. Mofid *, A. Vafai

Department of Civil Engineering, Sharif University of Technology, PO Box 11365-9313, Tehran, Iran

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Abstract

In this paper, a solution method is presented for the analysis of contact/impact between deformable bodies. The method is different from Lagrange multipliers and Penalty methods. Having defined a body as contactor and the other as target, it is assumed that contact forces are developed between contactor nodes and target surface. The method is based on stiffness transformation and eliminating the normal degree of freedom of contactor node. The method is absolutely general and can be used in static and dynamic nonlinear problems. All the drawbacks of Lagrange multipliers and Penalty method are vanished in this technique and it can be easily implemented in computer programs. The efficiency and applicability of the method is demonstrated by several illustrative examples. © 2000 Elsevier Science S.A. All rights reserved.

1. Introduction

The large amount of researches and efforts devoted to contact/impact problems during the last two decades reveals the importance of the phenomena. Various numerical methods and computational techniques have been proposed for the different classes of the problem, involving different nonlinearities due to the material property, finite geometry changes, or friction effects. However, a contact problem is inherently a nonlinear problem. The occurrence of contact can be at any location of contactor and target body and the basic condition of contact is that no material penetration may take place between the contacting bodies. Here, it is assumed that the contactor body contacts the target surface through its nodes. The present work deals with the analysis of general three dimensional contact problems which is applicable to both static and dynamic conditions, with all probable nonlinearities.

Lagrange Multipliers and Penalty methods are two widely used methods for treating contact/impact problems.

In the Lagrange multiplier method [2–4], the contact forces are Lagrange parameters and contact conditions are satisfied exactly. However, increase in the number of unknowns which are displacements and contact forces simultaneously, and change in the equation solution scheme are two disadvantages of the method. In the Penalty method [5–7], the contact forces are proportional to the amount of penetration by introducing a penalty number which is physically equivalent to an additional string between contacted bodies. The accuracy of solution is strongly dependent on the penalty number [7] and boundary conditions can only be approximately satisfied.

However, there are other methods which have been investigated as alternative solution techniques for contact problems and have shown good efficiency in special cases [9–12]. Depending on the problem,
choosing a suitable computational technique in nonlinear dynamics and a rate dependent or independent constitutive law and a proper plasticity rule may affect the accuracy and efficiency of the method [8].

The aim of this paper is to introduce a general numerical technique which can effectively direct the motion of a contactor node on target surface to meet the contact conditions. The loss of accuracy, increase in the number of unknowns, and modification in the solution scheme, which are major drawbacks of Penalty method and use of Lagrange Multipliers, are vanished in this method. The approach is based on transformation of contactor node degrees of freedom (dof) according to the target surface, and elimination of normal dof of the node. The contact forces are not computed directly. However, after the system is solved, they can be achieved from equilibrium equations. The method is absolutely general and can be used in static or dynamic analyses in presence of any kind of nonlinearities.

2. Contactor node and target surface

As shown in Fig. 1(a), consider a contactor element $C$ from a body contacting the surface of a target element $T$ as part of a target body, through a contactor node $P$. Assume that the coordinates of node $P$ on the target surface ABCD are obtained from a contact detection algorithm for which there are several methods. The condition is that there should be no overlap between two bodies. Since node $P$ is on the target surface, its coordinates at time $t$, that is $(x^P_1, x^P_2, x^P_3)$, can be explained in terms of target surface nodes coordinates as

$$x^P_i = \sum_{j=1}^{\text{nodes}} N_j x^j_i, \quad i = 1, 2, 3,$$

where $x^j_i$ is the coordinates of node $j$ of the surface in the direction $i$ at time $t$. $N_j$’s are shape functions evaluated at node $P$. Assume that $\vec{n}$ is normal to the surface at node $P$ as shown in Fig. 1(b).

The constraint on the motion of node $P$ is that as long as the reaction of node $P$ on the surface is compressional, it cannot move along the normal $\vec{n}$. We adopt the degrees of freedom of node $P$ in the direction of normal $\vec{n}$ (normal dof) and parallels to the surface $\vec{t}_1$ and $\vec{t}_2$, instead of the global coordinate system $XYZ$. This introduces simplicity in subsequent computations, namely the elimination of normal dof of the contactor node, because the constraint is to be imposed on the normal dof. Fig. 2(a) and (b) show the dof of $P$ before and after transformation.

Depending on the problem, the plane ABCD may be considered flat. Therefore, normal $\vec{n}$ can be taken as constant on the surface. However, in problems with higher order finite elements or significant geometrical changes, this is not a good approximation. Assuming a Lagrangian description of motion and isoparametric finite elements we have

$$x_i = x^0_i + u_i,$$
\[ \dot{x}_i^P = 0 \dot{x}_i^P + \ddot{u}_i^P = \sum_j \left[ N_j \left( 0 \dot{x}_j^P + \ddot{u}_j^P \right) \right], \]  
where left superscript shows the time, right superscript shows the node number and right subscript shows the direction.

\[ \dot{\xi}_j^P, \dot{\eta}_j^P, \dot{\zeta}_j^P; \]  

\[ \dot{N}_j \]  

\[ \hat{\mathbf{n}} = \hat{\mathbf{t}}_1 \times \hat{\mathbf{t}}_2 \]  

For the transformation of degrees of freedom of node \( P \) we need to introduce \( \tilde{\mathbf{t}}_1, \tilde{\mathbf{t}}_2, \tilde{\mathbf{n}} \). The calculations of vectors \( \tilde{\mathbf{t}}_1, \tilde{\mathbf{t}}_2, \tilde{\mathbf{n}} \), which define a proper orthonormal local coordinate system, can be performed exactly. However, there are some approximate methods [3] for the calculation of \( \tilde{\mathbf{n}} \) which offer more convenience and enough accuracy. The local transformation matrix \( [a] \) is defined as:

\[
\begin{bmatrix}
\tilde{\mathbf{t}}_1 \\
\tilde{\mathbf{t}}_2 \\
\tilde{\mathbf{n}}
\end{bmatrix} =
\begin{bmatrix}
t_{11} & t_{12} & t_{13} \\
t_{21} & t_{22} & t_{23} \\
n_{1} & n_{2} & n_{3}
\end{bmatrix}
\]  

Matrix \( [a] \) transforms the incremental displacements of node \( P \) (i.e. \( u_i^P \)) as

\[ \{ \tilde{\mathbf{u}}^P \} = [a] \{ \mathbf{u}^P \} \]  

or in component form

\[ u_i^P = \sum_{j=1}^{3} a_{ij} u_j^P, \quad i = 1, 2, 3, \]  

where \( \{ \mathbf{u}^P \} \) and \( \{ \tilde{\mathbf{u}}^P \} \) are the rotated and unrotated incremental displacement vectors of contactor node \( P \), respectively. Therefore, the transformation matrix \( [A] \) is defined as

\[
[A] =
\begin{bmatrix}
I_1 & 0 & 0 \\
0 & [a] & 0 \\
0 & 0 & I_2
\end{bmatrix},
\]  

where number of rows and columns of \( [A] \) is equal to the dof of contactor element \( C \). \( I_1 \) and \( I_2 \) are appropriate identity matrices and \( [a] \), as defined in Eq. (5), is set in the position of node \( P \). Therefore, we can write

\[ \left[ \dot{\mathbf{K}}^C \right] = [A] \left[ \dot{\mathbf{K}}^C \right] [A]^T \]  

or in component form

\[ \dot{K}_{ij}^C = \sum_r \sum_s A_{ir} A_{js} \dot{K}_{rs} \]  

Fig. 2. Contractor node degrees of freedom: (a) before rotation; (b) after rotation.
where \([{K^C}]\) and \([{K'^C}]\) are the original and transformed tangent stiffness matrices of contactor element \(C\) at time \(t\), respectively. The term tangent implies a general nonlinear problem. It is noted that the transformation process may be performed at the level of element or structure.

### 3. Uniting the contactor and target elements and eliminating the normal dof of the contactor node \(P\)

The movement of contactor node \(P\) must be compatible with the shape of target surface. To eliminate the normal dof of node \(P\), as shown in Fig. 2, the tangent stiffness matrix of elements \(C\) and \(T\) (that is \([{K^C}]\) and \([{K'^T}]\)) can be united as one stiffness matrix. Because, as long as node \(P\) is in contact with element \(T\), the dof of elements \(T\) and \(C\) are not independent. That is the reason we would like to call the method the United Elements. Since node \(P\) is on the surface ABCD shown in Fig. 2, the incremental displacement of node \(P\) in direction \(\bar{n}\), i.e. \(u_n^P\), can be explained as a function of \(\bar{n}\) and the incremental displacements of the surface nodes \(u_i^j\)'s as

\[
u_n^P = F(\bar{n}, u_i^j),
\]

where right superscript is the node and the right subscript shows the direction. It is noted that in writing Eq. (11), all the incremental displacements are infinitesimal.

After the contact takes place, a constraint arises. That is, the contactor node must not penetrate the target surface. A mathematical form of this constraint is the function \(F\) in Eq. (11) for infinitesimal incremental displacements. The problem is how to impose Eq. (11) to the motion of the system. It can be added to the system through a penalty number or use of Lagrange multipliers. However, there are other ways, such as treating the elements \(T\) and \(C\) as one element. Writing the static equilibrium equations of both elements simultaneously before contact, we have

\[
\begin{bmatrix}
[{K^C}] & 0 \\
0 & [{K'^T}]
\end{bmatrix}
\begin{bmatrix}
{u^C} \\
{u^T}
\end{bmatrix}
= \begin{bmatrix}
{f^C} \\
{f^T}
\end{bmatrix},
\]

where \(u\) and \(f\) are the incremental displacements and unbalanced force vectors, respectively. The right superscript show the corresponding element and the off-diagonal terms are zero which implies that the two elements are independent. However, after contact and imposing Eq. (11) nonzero off-diagonals will be generated. It is noted that since in a real nonlinear problem, the incremental displacements are finite, it is preferred to use Eq. (11) to define a single tangent stiffness matrix for the elements \(T\) and \(C\) as a united element. The concept of stiffness matrix helps to establish the stiffness matrix of the united element. That is, \(K_{ij}\) is the reaction at dof \(i\) due to unit displacement at dof \(j\) with other dof being restrained. Since the normal dof is to be eliminated, it has neither independent movement nor can be restrained. However, in order to avoid significant changes in stiffness matrix map and to avoid singularity, the off-diagonals of united stiffness matrix corresponding to normal dof are set to zero. After the system is solved, \(u_n^P\) cannot be obtained from Eq. (11) because this equation is written for infinitesimal incremental displacements. However, \(u_n^P\) can approximately be achieved from the final configuration of target surface and displacements of node \(P\) parallel to the surface. Assuming that node \(P\) is still on the surface and the reaction of the node is compressional, the position of node \(P\) on the target surface can be estimated from the final geometry of the system or equilibrium equations. The accuracy of the solution is improved by performing iterations to remove the unbalanced forces.

Example (1) manifests the details of this procedure clearly.

### 4. Illustrative examples

**Example 1.** Consider two 2D finite elements as shown in Fig. 3. Their tangent stiffness matrices are \([K^1]\) and \([K^2]\), respectively, where left superscript shows the time and right superscript shows the element number.
Fig. 3 (a) shows the elements dof before contact; (b) elements dof after contact; (c) dof of the united element with eliminated normal dof.

Fig. 3 (a) shows the elements dof and Fig. 3 (b) shows node numbers and rotated dof of the contactor node number 6 according to the target surface 1–2. The nodes have only translational displacements. It is desired to find a single stiffness matrix for the elements. To unite the elements, it is necessary to eliminate the normal dof number 12 shown in Fig. 3(c).

The element shape functions are:

\[
N_1 = \frac{1}{4} (1 + \xi)(1 + \eta); \quad N_2 = \frac{1}{4} (1 - \xi)(1 + \eta),
\]
\[
N_3 = \frac{1}{4} (1 - \xi)(1 - \eta); \quad N_4 = \frac{1}{4} (1 + \xi)(1 - \eta).
\]

The rotated stiffness matrix of element number 2 is \( T^T K_2 T \) where

\[
T = \begin{bmatrix}
1 & 1 & \cos \alpha & - \sin \alpha \\
& 1 & \sin \alpha & \cos \alpha \\
& & \ddots & \\
& & & 1
\end{bmatrix}
\]

and \( \alpha \) is shown in Fig. 3(b). Let the stiffness matrix of the united element be \( \hat{K} \). From the definition of stiffness matrix, \( \hat{K}_{ij} \) is the reaction at dof \( i \) due to a unit displacement at dof \( j \) with other freedoms being restrained. The important point is that since the normal dof number 12 shown in Fig. 3(c) is to be eliminated, it cannot be restrained.

As an example Fig. 4 demonstrates how to calculate \( \hat{K}_{11} \). The reactions of supports are given in the first column of \( \hat{K}_{ij} \) given below.

Thus the united stiffness matrix \( \hat{K} \) can be obtained from the definition of \( \hat{K}_{ij} \) as

\[
C_1 = \frac{a}{L} \sin \alpha, \quad C_2 = \frac{a}{L} \cos \alpha,
\]
\[
C_3 = \left(1 - \frac{a}{L}\right) \sin \alpha, \quad C_4 = \left(1 - \frac{a}{L}\right) \cos \alpha,
\]
The position of contactor node \( P \) is solved. The position of contactor node \( P \) is computed from equilibrium equations or the displacement of dof number 11 and final position of the target does not change. After the equilibrium equations are solved, the answer for normal dof is ignored and it is not increased and the solution scheme does not change. Therefore, it is independent of other degrees of freedom. Since the diagonal member is not zero, the stiffness matrix is not singular. Therefore, it is independent of other degrees of freedom. Since the diagonal member is not zero, the stiffness matrix is not singular.

\[
\begin{bmatrix}
K_{11} + C_1^2 K_{14}^n \\
K_{12}^n + C_1 C_2 K_{14}^n \\
K_{13}^n + C_1 C_3 K_{14}^n \\
K_{14}^n + C_1 C_4 K_{14}^n \\
K_{21} + C_2 C_1 K_{24}^n \\
K_{22} + C_2 C_2 K_{24}^n \\
K_{23} + C_2 C_3 K_{24}^n \\
K_{24} + C_2 C_4 K_{24}^n \\
K_{31} + C_3 C_1 K_{34}^n \\
K_{32} + C_3 C_2 K_{34}^n \\
K_{33} + C_3 C_3 K_{34}^n \\
K_{34} + C_3 C_4 K_{34}^n \\
K_{41} + C_4 C_1 K_{44}^n \\
K_{42} + C_4 C_2 K_{44}^n \\
K_{43} + C_4 C_3 K_{44}^n \\
K_{44} + C_4 C_4 K_{44}^n
\end{bmatrix}
\]

which is symmetric. As can be seen, the off-diagonal members of dof 12 are intentionally set to zero. Therefore, it is independent of other degrees of freedom. Since the diagonal member is not zero, the stiffness matrix is not singular. Therefore, the number of unknowns is not increased and the solution scheme does not change. After the equilibrium equations are solved, the answer for normal dof is ignored and it is computed from equilibrium equations or the displacement of dof number 11 and final position of the target surface 1–2. Fig. 5 depicts the approximate situation of contactor node as discussed above. The final position of the target surface at time \( t + \Delta t \) and the increment in dof number 11, \( u_{11} \), are achieved after the system is solved. The position of contactor node \( P \), is obtained from the intersection point of a line parallel to \( \hat{n} \) (normal to the target surface at time \( t \)), and target surface at time \( t + \Delta t \).
The process of uniting can be performed at the level of structure instead of element to minimise the numerical effort and ease of computer programming. Also the procedure can be employed for any number of contactor nodes.

Example 2. Two bodies shown in Fig. 6 are pushed against each other. The Total Lagrangian (TL) formulation is used for geometric nonlinearities. When using the TL formulation, special attention must be paid to large strains, and it is based on ignoring the second and third order terms of incremental displacement in virtual work equations [13,14]. The bodies are discretized to 4 node plane strain elements. The contact is static and frictionless.

As shown in Fig. 6 the initial position of contactor nodes are at the middle of target elements and sliding of contactor nodes has occurred without friction.

Example 3. This example shows a special case of contact-impact problem, that is deformable-rigid case. As demonstrated in Fig. 7, a ring collides an inclined frictionless rigid surface. The ring is built from 16 solid elements and the inner and outer radii are 20 and 30, respectively. The normal reaction of rigid surface is calculated based on linear theory and also including geometric nonlinearity. The TL formulation is used and the contact-impact is frictionless. According to the united element procedure, the degrees of freedom of contactor node must rotate according to the rigid surface. The dof normal to the rigid surface must be restrained by setting all the off-diagonal members of this dof equal to zero [1]. After the solution of dynamic equilibrium equations, the incremental displacement of normal dof must be set to zero, because no penetration must take place. In Fig. 7, \( V \) is initial velocity and \( M \) is mass and subscripts show the direction. \( E \) and \( v \) are modulus of elasticity and Poisson’s ratio, respectively.
Fig. 8 shows the normal reaction of the rigid surface with and without geometric nonlinear effects. The comparison between the efficiency of the methods reveals that for linear analysis of the above example, the united element method is approximately 1.7 times faster than the Lagrange multipliers method.

5. Conclusions

A numerical method is presented in detail for body to body contact/impact problems. The method is absolutely general and can be used with all kinds of nonlinearities. It has not the drawbacks of Lagrange multipliers family and Penalty methods. That is, the number of unknowns does not increase and the solution scheme needs no modification. Besides, the boundary conditions are satisfied exactly. However, it needs some matrix transformation and modification in the structure or elements stiffness matrix. The idea is to unite the contactor and target elements. Illustrative examples are given to demonstrate the capability and efficiency of the method.

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References